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Confined solutions of multidimensional inversion equations

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Abstract. We establish an inversion equation associated to a system of n linear first-order partial differential equations with q different variables coordinates, $2 \leqslant q \leqslant n$ (the partial differential linear operator being a diagonal matrix).

Contrary to the one-dimensional case, the degenerate kernels of the multidimensional inversion equations are not necessarily of the pure exponential type and an infinite number of other functions is possible. We get for $q = 2$ that, among the potentials corresponding to these degenerate kernels there exist confined ones in the two-dimensional coordinate plane. For $q \geqslant 3$, the potentials are not entirely confined in the whole coordinate space. Moreover for $q \geqslant 3$, the reconstructed potentials must satisfy well defined non-linear equations. As an application if, for $n = 3$, we interpret one coordinate as a time and the other two as spatial coordinates, then the non-linear three-wave evolution equation exhibits an infinite number of particular solutions which, for any finite time, are confined in the coordinate plane.

1. Introduction

There is actually a great interest (Novikov 1977) in the explicit construction of simple really confined solutions of non-linear multidimensional evolution equations. A preliminary problem is the existence (or not) of confined solutions in the multidimensional coordinate space. In the inverse scattering framework, practically this means that we have to find out whether degenerate kernels of the inversion equation (IE) can lead to reconstructed potentials vanishing asymptotically in all directions of coordinate space. If the kernel of the IE depends in fact upon only one variable then these degenerate kernels are purely exponential (as is the case for the IE with one coordinate), and confined solutions are possible in the one-dimensional space R and not possible in higher-dimensional coordinate space R^q, $q > 1$. Recently, an IE associated to a partial differential system with pure exponential degenerate kernels has been established (Cornille 1978). In this paper, for the same system, we show that there exists also another IE where the kernels belong to a more general class and lead to a progress concerning the confinement problem.

Let us state the problem that we consider. We start from an $n \times n$ linear partial differential system:

$$(\Delta_0(x_1, x_2, \ldots, x_n) + ik\Lambda - Q(x_1, \ldots, x_n))\psi(x_1, \ldots, x_n) = 0 \quad (1)$$

where Λ is a diagonal eigenvalue matrix, $\Lambda=(\delta_{ij}\lambda_i)$; $Q(x_1,\ldots,x_n)$ is an $n\times n$ 'potential'

$$Q=\begin{pmatrix} q_1^1 & \ldots & q_1^n \\ \vdots & & \vdots \\ q_n^1 & \ldots & q_n^n \end{pmatrix}$$

and ψ is a column vector. Δ_0 is a diagonal matrix partial differential operator $\Delta_0=(\delta_{ij}\mu_i(x_i)\partial/\partial_{x_i})$ where each element $\mu_i(x_i)\partial/\partial_{x_i}$ operates on a particular coordinate x_i and any two elements operate on two different coordinates; $\delta_{ii}=1$, $\delta_{ij}=0$, $i\neq j$, μ_i being positive arbitrary functions such that $\lim_{x_i\to\infty}\int_{\text{constant}}^{x_i}\mu_i^{-1}(u)\,du=+\infty$. Let us define $\bar{x}_i=\int^{x_i}\mu_i^{-1}(u)\,du$, then equation (1) can be rewritten:

$$(\Delta_0(\bar{x}_1,\ldots,\bar{x}_n)+ik\Lambda-Q(\bar{x}_1,\ldots,\bar{x}_n))\psi(\bar{x}_1,\ldots,\bar{x}_n)=0$$

where now $\Delta_0(\delta_{ij}\partial/\partial_{\bar{x}_i})$. It follows that we do not reduce the generality of our system by choosing $\mu_i\equiv 1$ in equation (1) and we reduce our study to this case in the following.

Let us consider a set of solutions

$$(D_0(x_i)+i\lambda_i k)U_i^0(x_i)=0, \qquad D_0(x)=\partial/\partial_x$$

and a set of eigenfunctions $\{\psi_j^0=(\delta_{ij}U_j^0(x_j))\}$ when $Q\equiv 0$:

$$(\Delta_0+i\Lambda k)(\psi_1^0,\psi_2^0,\ldots,\psi_n^0)=(0).$$

Let us formally define a set of n column vectors $\{\psi_j\}$, each with n components, $i=1,\ldots,n$:

$$\psi_j=\left(\delta_{ij}U_j^0(x_j)+\int_{x_j}^{\infty}U_j^0(y)K_i^j(x_1,\ldots,x_n;y)\,dy\right) \tag{2}$$

that we would like to be eigenfunctions when $Q\not\equiv 0$: $(\Delta_0+ik\Lambda-Q)(\psi_1,\psi_2,\ldots,\psi_n)=(0)$.

In order to get this result we recall that if the transform $\{K_j^i\}$ are such that the representation (2) exists and

$$\lim_{y\to\infty}U_i^0(y)K_j^i(x_1,\ldots,x_n;y)=0 \tag{3}$$

$$O_{j\times j}^iK_j^i=\sum_{m\neq j}^{m}K_m^i\hat{K}_j^m\lambda_j(\lambda_m)^{-1}$$

$$O_{j\times j}^i=D_0(x_j)+\lambda_j(\lambda_i)^{-1}D_0(y), \qquad \hat{K}_j^m=K_j^m(x_1,\ldots,x_n;y=x_m) \tag{4}$$

$$q_j^i=\lambda_j(\lambda_i)^{-1}\hat{K}_j^i, \qquad q_i^i=0, \tag{5}$$

where as above $D_0(u)=\partial/\partial u$, then one can show that the $\{\psi_j\}$ are solutions of equation (1) (Cornille 1978).

Our aim is to seek an integral equation, that we will call the inversion equation (IE), such that the solutions $\{K_j^i\}$ satisfy the non-linear partial differential equations (NLPDE) (equation (4)) and where the degenerate kernels will not be purely exponentials.

In § 2 we derive such an IE where the kernels F_j^i depend upon two variables, say s, y, and n parameters $x_1,\ldots,x_n$ subject to only n linear partial differential equations (LPDE). In order to see clearly the progress we compare with the previous one-coordinate case (Cornille 1977) or with our previous IE. In these cases F_j^i depends upon only one variable $U_j^i = \lambda_j s - \lambda_i y$ and the degenerate kernels (function of s multiplied by functions of y) can only be pure exponentials.

In the present paper, F_j^i depends in fact upon two independent variables

$$U_j^i = \lambda_j(x_j - s) + \sum_{m \neq j} \lambda_m x_m, \qquad V_j^i = \lambda_i(x_i - y) + \sum_{m \neq i} \lambda_m x_m,$$

such that the degenerate kernels are not restricted to be purely exponential. We can take for instance some exponentially decreasing function of $(U_j^i)^2$ multiplied by a similar function of $(V_j^i)^2$ both vanishing when $|U_j^i| \to \infty$ or $|V_j^i| \to \infty$ (Gaussians, ...). For the reconstructed potentials, the products of such functions appear in the numerator while the denominator can be bounded (for this discussion we do not consider the cases where the Fredholm determinants can vanish). For $n = 2$, the independent variables associated with U_j^i and V_j^i are finally $\lambda_i x_i$ and $\lambda_j x_j (i \neq j)$ such that the potentials can vanish asymptotically in the whole x_1, x_2 plane. For $n = 3$ we get the product of functions with $\lambda_i x_i + \lambda_j x_j$ as variables and so while this is a step forward compared with our previous pure exponential-type kernels (Cornille 1978), there still exist in the x_1, x_2, x_3 space, directions where the solutions are not confined. Similarly for $n = 4, 5, \ldots$ our solutions are not confined in the whole coordinate space.

Further, we find for $n \geq 3$ that our reconstructed potentials must satisfy well defined extra NLPDE (besides those of equation (4)) in such a way that our inversion formalism applies in fact only in a subspace of the whole space of possible potentials associated with the system (1).

In § 3 we derive an IE associated with the $n \times n$ system (1) when the coordinate space is R^q and Δ_0 is such that $x_q = x_{q+1} = \ldots = x_n$, $(q \geq 2)$. The results are very similar to the previous ones if we replace n by q. For $q = 2$ there exist confined potentials and not for $q \geq 3$. Further, for $q \geq 3$, the potentials must also satisfy well defined extra NLPDE. If for $q = 3$ we interpret one coordinate as a time and the other two as spatial coordinates then these NLPDE have for any finite time, solutions which are confined in R^2.

In § 4 we study more particularly this possibility for $n = q = 3$. Considering in equation (1) and in our IE that one x_i is a time while the other two x_j, x_k are coordinates, then the above NLPDE can be interpreted as a non-linear three-wave evolution equation and we show explicitly the existence of confined solutions in R^2 for any finite time.

2. Inversion equations associated with the system (1) when the number of different coordinates is N

2.1. Integral equation

Let us consider the following integral equation:

$$K_j^i(x_1, \ldots, x_n; y) = \tilde{F}_j^i(x_1, \ldots, x_n; y) + \sum_m^{m=n} \int_{x_m}^{\infty} F_m^i(s; x_1, \ldots, x_n; y) K_j^m(x_1, \ldots, x_n; s)\, \mathrm{d}s \tag{6}$$

$$\tilde{F}_j^i(x_1, \ldots, x_n; y) = F_j^i(s = x_j; x_1, \ldots, x_n; y).$$

We remark that the free term $\tilde{F}^i_j$ is the restriction when $s=x_j$ of the kernel F^i_j. For each kernel $F^i_j(s; x_1, \ldots, x_n; y)$ we assume the boundary condition

$$\lim_{y\to\infty} F^i_j = 0, \qquad \lim_{s\to\infty} F^i_j K^j_l(x_1, \ldots, x_n; s) = 0, \tag{7}$$

and that they satisfy n independent LPDE,

$$\lambda_1^{-1} D_0(x_1) F^i_j = \lambda_2^{-1} D_0(x_2) F^i_j = \ldots = \lambda_n^{-1} D_0(x_n) F^i_j = -(\lambda_j^{-1} D_0(s) + \lambda_i^{-1} D_0(y)) F^i_j \tag{8}$$

and we assume also of course that the solution of equation (6) exists and is unique.

Property. If we assume that the $\{F^i_j\}$ satisfy both equations (7) and (8), then the solutions $\{K^i_j\}$ of equation (6) satisfy the NLPDE (equation (4)). For the proof let us remark that due to equation (8), $O^i_{jx_j}\tilde{F}^i_j = 0$ and applying $O^i_{jx_j}$ to both sides of equation (6):

$$O^i_{jx_j} K^i_j = -\tilde{F}^i_j \hat{K}^j_j + \sum_m \int F^i_m D_0(x_j) K^m_j + \sum_m \int K^m_j O^i_{jx_j} F^i_m.$$

Taking into account relations (7) and (8), the right-hand side can be written

$$\sum_{m\neq j} \frac{\lambda_j}{\lambda_m} \tilde{F}^i_m \hat{K}^m_j + \sum_m \int F^i_m O^m_{jx_j} K^m_j \, ds$$

and comparing with the solution of equation (6), the result (equation (4)) follows from the uniqueness assumption of the solution of equation (6).

In conclusion if the kernels $\{F^i_j\}$ satisfy equations (7) and (8), if we substitute the solutions $\{K^i_j\}$ of equation (6) into the representation (2), if further the condition (3) is satisfied, then equations (2) are solutions of our starting partial differential system (1) and consequently equation (6) will be an associated IE such that $q^i_j = (\lambda_j/\lambda_i)\hat{K}^i_j$, $q^i_i = 0$.

Let us define

$$\mathcal{K}(x_1, \ldots, x_n; y) = (K^i_j(x_1, \ldots, x_n; y)) = \begin{pmatrix} K^1_1 & \ldots & K^1_n \\ \vdots & & \vdots \\ K^n_1 & \ldots & K^n_n \end{pmatrix}$$

$$\tilde{\mathcal{F}}(x_1, \ldots, x_n; y) = (\tilde{F}^i_j(x_1, \ldots, x_n; y)),$$

$$\mathcal{F}(s; x_1, \ldots, x_n; y) = (F^i_j(s; x_1, \ldots, x_n; y)\theta(s-x_j))$$

where θ is the Heaviside distribution, then equation (6) can be written in a matrix form:

$$\mathcal{K}(x_1, \ldots, x_n; y) = \tilde{\mathcal{F}}(x_1, \ldots, x_n; y) + \int_{-\infty}^{+\infty} \mathcal{F}(s; x_1, \ldots, x_n; y)\mathcal{K}(x_1, \ldots, x_n; s)\, ds \tag{6a}$$

2.2. *Properties of the kernels* F^i_j

Let us first assume that the kernels F^i_j are *independent of the coordinates* $x_1, \ldots, x_n$; i.e. $F^i_j = F^i_j(s; y)$, $\tilde{F}^i_j = F^i_j(x_j; y)$. From $\partial F^i_j/\partial x_j = 0$ we see that equation (8) reduces to $(D_0(s) + \lambda_j(\lambda_i)^{-1} D_0(y))F^i_j = 0$ and $O^i_{jx_j}\tilde{F}^i_j = 0$, and equation (6) reduces to our previous IE (Cornille 1978) associated with the system (1). (If further $x_1 = x_2 = \ldots = x_n$, these LPDE are identical to those for the one-coordinate case.)

In fact F_j^i depends upon only one variable U_j^i:

$$F_j^i(U_j^i = \lambda_j s - \lambda_i y). \tag{9}$$

The only possible degenerate kernels are of the exponential type $\exp[-\gamma_m(\lambda_j s - \lambda_i y)]$ (or superposition of such terms). If further $\lambda_i \lambda_j > 0$, these kernels cannot go to zero when $s \to \infty$ and $y \to \infty$, which leads to difficulties concerning the existence of the solutions of equation (6) (for instance we must necessarily take $F_i^i \equiv 0$).

Now we assume that the F_j^i *depend upon the* $x_1, \ldots, x_n$. The main advance obtained here is that in order to satisfy the LPDE (equation (8)), F_j^i can now be written as a function of two independent variables U_j^i and V_j^i:

$$F_j^i\left(U_j^i = \lambda_j(x_j - s) + \sum_{m \neq j}^{m=n} \lambda_m x_m + \nu_j^i;\ V_j^i = \lambda_i(x_i - y) + \sum_{m \neq i}^{m=n} \lambda_m x_m + \eta_j^i\right) \tag{10}$$

where ν_j^i and η_j^i are constants. If we consider degenerate kernels $F_j^i = g_j^i(U_j^i)h_j^i(V_j^i)$ (or superposition $\Sigma\, g_{j,m}^i h_{j,m}^i$) we see that a larger class of functions exists such that all the above derivations are correct.

(i) Let us still consider exponential-type kernels

$$F_j^i = \sum_m \beta_{j,m}^i \exp[-\gamma_{j,m}^i(U_j^i + \delta_{j,m}^i V_j^i)]$$

where $\delta_{j,m}^i$ and $\gamma_{j,m}^i$ are arbitrary constants. We can always choose the signs of $\delta_{j,m}^i$ and $\gamma_{j,m}^i$ in such a way that $F_j^i \to 0$ when either $s \to \infty$ or $y \to \infty$.

(ii) In principle, we can choose for g_j^i, h_j^i any kind of functions decreasing sufficiently when $s \to \infty$, $y \to \infty$, so that both the solution of equation (6) exists and all the sufficient conditions (equations (3), (7) and (8)) are satisfied. However we focus our attention on reconstructed potentials $\hat{K}_j^i$ which could be confined in R^n and so we will exhibit simple examples where this problem is easy to study. Let us consider for instance:

$$\begin{aligned} &g_j^i(U_j^i) = (U_j^i)^{m_0} \exp[-(U_j^i)^{2m_1}] \\ &h_j^i(U_j^i) = (V_j^i)^{m_2} \exp[-(V_j^i)^{2m_3}] \\ &m_0 > 0, \qquad m_2 > 0, \qquad m_1 > 0 \text{ integer}, \qquad m_3 > 0 \text{ integer} \end{aligned} \tag{11}$$

and for F_j^i a sum of a finite number of such terms.

In order to have a crude insight of what can happen, let us first remark that $\hat{K}_j^i = \tilde{F}_j^i(y = x_i) + \text{other terms}$ and investigate $\tilde{F}_j^i(y = x_i)$ in the simplest case $F_j^i = g_j^i h_j^i$:

$$\tilde{F}_j^i(y = x_i) = g_j^i\left(\sum_{m \neq j}^{m=n} \lambda_m x_m + \nu_j^i\right) h_j^i\left(\sum_{m \neq i}^{m=n} \lambda_m x_m + \eta_j^i\right), \qquad \text{for } i \neq j. \tag{12}$$

For $n = 2$ we have the product $g_j^i(\nu_j^i + \lambda_i x_i) h_j^i(\eta_j^i + \lambda_j x_j)$. When $\rho = (\Sigma\, x_i^2)^{1/2} \to \infty$, either $|x_1| \to \infty$ or $|x_2| \to \infty$ (or both $\to \infty$) and necessarily we get for equation (11)-type kernels that $|\tilde{F}_j^i(y = x_i)| \to 0$.

For $n = 3$, we have the product $g_j^i(\nu_j^i + \lambda_i x_i + \lambda_k x_k) h_j^i(\eta_j^i + \lambda_j x_j + \lambda_k x_k)$ such that in R^3, there exist asymptotic directions where $|\tilde{F}_j^i| \not\to 0$ when $\rho \to \infty$. The same conclusion holds for $n > 3$.

2.3. Simple examples

2.3.1. For simplicity we first assume that the kernels of equation (6) are of the most simple form $F^i_j = g^i_j h^i_j$ given by equation (11).

For $n = 2$ we consider first $F^i_i \equiv 0$ and

$$\mathscr{F} = \begin{pmatrix} 0 & F^1_2\theta(x_2 - s) \\ F^2_1\theta(x_1 - s) & 0 \end{pmatrix} \tag{13a}$$

and get for $(i, j) = (1, 2)$ and $(2, 1)$:

$$D\hat{K}^i_j = g^i_j(\nu^i_j + \lambda_i x_i)h^i_j(\eta^i_j + \lambda_j x_j) = \tilde{F}^i_j(y = x_i)$$
$$D = 1 - A^1_{12}A^2_{21}, \qquad A^i_{ij} = \int_0^\infty g^i_j(-\lambda_j u + \lambda_i x_i + \nu^i_j)h^i_j(-\lambda_j u + \lambda_i x_i + \eta^i_j)\, du. \tag{13b}$$

D is bounded when $\rho = (x_1^2 + x_2^2)^{1/2} \to \infty$, $D\hat{K}^i_j \to 0$ and finally the potentials $q^i_j = \lambda_j(\lambda_i)^{-1}\hat{K}^i_j (i \neq j)$ are confined in the x_1, x_2 plane. (In all the discussion in this paper we do not consider the cases where D vanishes). Secondly we introduce $F^i_i\theta(x_i - s)$, $(i = 1, 2)$ in (13a); $F^i_j = g^i_i h^i_i$ and get:

$$D\hat{K}^2_1 = [g^2_1 h^2_1(1 - A^1_{11}) + h^2_1 g^1_1 A^2_{11}](A^2_{22} - 1) - h^2_2[g^1_1 A^2_{11} A^2_{12} + g^2_1 A^2_{12}(1 - A^1_{11})]$$
$$\text{if } 1 \leftrightarrow 2, \text{ then } \hat{K}^2_1 \leftrightarrow \hat{K}^1_2$$
$$D = (A^2_{21} - 1)(1 - A^1_{11}) + [A^1_{12}(A^2_{22} - 1) - A^1_{22}A^2_{12}][A^2_{21}(A^1_{11} - 1) - A^1_{21}A^2_{11}]$$

where

$$g^i_i = g^i_i(\lambda_j x_j + \nu^i_i), \qquad h^i_i = h^i_i(\lambda_j x_j + \eta^i_i),$$
$$g^i_j = g^i_j(\lambda_i x_i + \nu^i_j), \qquad h^i_j = h^i_j(\lambda_j x_j + \eta^i_j).$$

Further

$$A^i_{jk} = \int_0^\infty g^i_k h^k_j \, du$$

where

$$g^i_k = g^i_k(-\lambda_k u + \lambda_i x_i + \nu^i_k) \qquad \text{if } i \neq k;$$
$$g^i_k = g^i_k(-\lambda_i u + \lambda_j x_j + \nu^i_i) \qquad \text{if } i = k \neq j$$

and

$$h^k_j = h^k_j(-\lambda_k u + \lambda_j x_j + \eta^k_j) \qquad \text{if } j \neq k,$$
$$h^k_j = h^k_j(-\lambda_j u + \lambda_i x_i + \eta^j_j) \qquad \text{if } k = j \neq i.$$

When $\rho \to \infty$, D and $A^i_{j,k}$ are bounded and $g^i_j h^i_j$, $g^j_j h^i_j$, $g^j_j h^j_i$, $g^i_i h^i_j \to 0$. It follows for $D \neq 0$ that $\hat{K}^1_2$ and $\hat{K}^2_1$ are confined in the x_1, x_2 plane.

For $n = 3$ we consider two examples of degenerate kernels:

(i)

$$\mathscr{F} = \begin{pmatrix} 0 & F^1_2\theta(x_2 - s) & 0 \\ 0 & 0 & F^2_3\theta(x_3 - s) \\ F^3_1\theta(x_1 - s) & 0 & 0 \end{pmatrix} \tag{14a}$$

$$\hat{\mathscr{K}} = (\hat{K}^i_j),$$

$$D\hat{\mathscr{K}} = \begin{pmatrix} h^1_2 & 0 & 0 \\ 0 & h^2_3 & 0 \\ 0 & 0 & h^3_1 \end{pmatrix} \begin{pmatrix} A^1_{32}A^2_{13} & 1 & A^1_{32} \\ A^2_{13} & A^2_{13}A^3_{21} & 1 \\ 1 & A^3_{21} & A^3_{21}A^1_{32} \end{pmatrix} \begin{pmatrix} g^3_1 & 0 & 0 \\ 0 & g^1_2 & 0 \\ 0 & 0 & g^2_3 \end{pmatrix} \tag{14b}$$

$$D = 1 - A^3_{21}A^1_{32}A^2_{13}$$

(ii)

$$\mathscr{F} = \begin{pmatrix} 0 & F^1_2\theta(x_2-s) & F^1_3\theta(x_3-s) \\ F^2_1\theta(x_1-s) & 0 & 0 \\ F^3_1\theta(x_1-s) & 0 & 0 \end{pmatrix} \tag{15a}$$

$$\begin{aligned} &D\hat{K}^2_1 = h^2_1[g^2_1 + A^1_{13}(g^3_1A^2_{31} - g^2_1A^3_{31})], && \hat{K}^2_1 \leftrightarrow \hat{K}^3_1 \quad \text{if } 2\leftrightarrow 3 \\ &D\hat{K}^1_3 = g^1_3[h^1_3 + A^1_{12}(h^1_2A^2_{31} - h^1_3A^2_{21})], && \hat{K}^1_3 \leftrightarrow \hat{K}^1_2 \quad \text{if } 3\leftrightarrow 2 \\ &D\hat{K}^2_3 = h^2_1g^1_3A^2_{31}, && \hat{K}^2_3 \leftrightarrow \hat{K}^3_2 \quad \text{if } 3\leftrightarrow 2 \\ &D = (1 - A^1_{12}A^2_{21})(1 - A^3_{31}A^1_{13}) - A^1_{13}A^1_{12}A^2_{31}A^3_{21} \end{aligned} \tag{15b}$$

where in both (14*b*), (15*b*)

$$\begin{aligned} &h^i_j = h^i_j(\lambda_jx_j + \lambda_kx_k + \eta^i_j), \qquad g^i_j = g^i_j(\lambda_ix_i + \lambda_kx_k + \nu^i_j) \\ &A^i_{jk} = \int_0^\infty g^i_k(-\lambda_ku + \lambda_ix_i + \lambda_jx_j + \nu^i_k)h^k_j(-\lambda_ku + \lambda_ix_i + \lambda_jx_j + \eta^k_j)\,\mathrm{d}u. \end{aligned} \tag{15c}$$

In both examples (14*a*, *b*), (15*a*, *b*), while D remains bounded when $\rho = (x_1^2 + x_2^2 + x_3^2)^{1/2} \to \infty$, there remain asymptotic directions in the x_1, x_2, x_3 space where $D\hat{K}^i_j \nrightarrow 0$. However, if we consider finite values for one coordinate, let us say x_k, then the $\hat{K}^i_j$ are confined in the x_i, x_j plane.

2.3.2. Let us assume secondly that the degenerate kernels F^i_j are a finite sum of kernels of the type (11):

$$F^i_j = \sum_{m=1}^{m=m_0} g^i_{j,m}h^i_{j,m}.$$

For $n = 2$ and $\mathscr{F}$ given by (13*a*), even if the solutions are too complicated to be written in closed form (as in (13*b*) when $m_0 = 1$), we can show the confinement properties. We get for $(i, j) = (1, 2)$ and $(2, 1)$:

$$\begin{aligned} &\hat{K}^i_j = \sum_m h^i_{j,m}\Big(g^i_{j,m} + \sum_l B^i_{j,l}C^i_{m,l}\Big) \\ &B^i_{j,l}\Big(1 - \sum_m C^j_{l,m}C^i_{m,l}\Big) - \sum_{l'} B^i_{j,l'} \sum_m C^i_{m,l'}C^j_{l,m} = \sum g^i_{j,m}C^j_{l,m} \end{aligned} \tag{13c}$$

where

$$g^i_{j,m} = g^i_{j,m}(\lambda_ix_i + \nu^i_{j,m}), \qquad h^i_{j,m} = h^i_{j,m}(\lambda_jx_j + \eta^i_{j,m})$$

$$C^i_{m,l} = \int_0^\infty g^i_{i,m}(-\lambda_iu + \lambda_jx_j + \nu^i_{j,m})h^i_{j,l}(-\lambda_iu + \lambda_jx_j + \eta^i_{j,l})\,\mathrm{d}u.$$

Excluding the zeros of the Fredholm determinants we see that the $C^i_{m,l}$ and the $B^i_{j,l}$ are bounded when $\rho=(x_1^2+x_2^2)^{1/2}\to\infty$. For $\hat{K}^i_j$ always appear in (13c) the factors $h^i_{j,m}g^i_{j,m}$ or $h^i_{j,m}g^i_{j,l}$ which vanish when $\rho\to\infty$ and consequently the $\hat{K}^i_j\to 0$ when $\rho\to\infty$.

2.3.3. We consider $n=2$, $F^i_j(i\neq j)$ given by (13a) and pure exponential degenerate kernels. If the F^i_j are independent of x_1, x_2, $\hat{K}^i_j$ depend upon only one variable $\lambda_1x_1-\lambda_2x_2$ and so cannot be confined (Cornille 1978). When F^i_j depend also upon x_1, x_2, we give examples where the solution is not confined.

(i) We consider in (13a, b) the most degenerate case $F^i_j=a^i_j\exp[\nu^i_j(\lambda_j(x_j-s)+\lambda_ix_i)+\eta^i_j(\lambda_i(x_i-y)+\lambda_jx_j)]$, Re $\nu^i_j>0$, Re $\eta^i_j>0$; a^i_j, ν^i_j, η^i_j *being fixed constants* and get:

$$\hat{K}^i_j=\frac{a^i_j\exp\xi_i}{1-c\exp(\xi_1+\xi_2)}$$

where

$$\xi_i=\nu^i_j\lambda_ix_i+\eta^i_j\lambda_jx_j, \qquad i=1,2$$
$$c=a^1_2a^2_1[\lambda_1\lambda_2(\nu^1_2+\eta^2_1)(\nu^2_1+\eta^1_2)]^{-1} \tag{13b'}$$

and Re ξ_1, Re ξ_2 are two independent directions of the x_1, x_2 plane (Re$(\nu^1_2\nu^2_1-\eta^1_2\eta^2_1)\neq 0$). In the x_1, x_2 plane let us consider asymptotic directions along Re $\xi_i=0$ where necessarily $|\text{Re}\ \xi_j|\to\infty$. When Re $\xi_j\to+\infty$ then $|\hat{K}^i_j|\to 0$, however when Re $\xi_j\to-\infty$, then $|\hat{K}^i_j|\to|a^i_j|\neq 0$ and the potentials *are not confined.* If further we require $\hat{K}^i_j=$ constant $(\hat{K}^j_i)^*$ or constant $\hat{K}^j_i$, at least either $\xi_i=\xi^*_j$ or $\xi_i=\xi_j$ and $|\hat{K}^i_j|$ depends upon only one variable Re $\xi_i=$ Re ξ_j and is not confined.

(ii) We consider in (13a) and (13c) $m_0=2$, for simplicity we take $\lambda_1=+1$, $\lambda_2=-1$ and reduce our study to $\hat{K}^2_1=\rho(\hat{K}^1_2)^*$ ($\hat{K}^2_1$ complex) or $\hat{K}^2_1=\rho\hat{K}^1_2$ ($\hat{K}^2_1$ real) with ρ real. For the F^i_j we take

$$F^1_2=\sum_{m=1}^{2}a_m\exp[\eta_{1,m}(s-x_2+x_1)+\eta_{2,m}(y-x_1+x_2)],$$

$$F^2_1=\sum_{m=1}^{2}a^*_m\exp[\eta^*_{2,m}(s-x_1+x_2)+\eta^*_{1,m}(y-x_1+x_2)],$$

$$\text{Im}\ \eta_{i,j}=\text{Im}\ \eta_{i,i}, \qquad \text{Re}\ \eta_{i,j}>0,$$

a_m and $\eta_{n,m}$ being fixed constants. We find

$$-\rho|\hat{K}^1_2|^2=(\ln D)_{x_1x_2}=ND^{-2}$$

where D is the Fredholm determinant of equation (6).

$$D=1+c_1\,e^{2\xi_1}+c_2\,e^{2\xi_2}+c_3\,e^{\xi_1+\xi_2}+c_4\,e^{2(\xi_1+\xi_2)}$$
$$N=c_5\,e^{2\xi_1}+c_6\,e^{2\xi_2}+c_7\,e^{\xi_1+\xi_2}+c_8\,e^{2(\xi_1+\xi_2)}+c_9\,e^{3\xi_1+\xi_2}+c_{10}\,e^{3\xi_2+\xi_1}$$

$\xi_i=x_1$ Re $\eta_{1,i}+x_2$ Re η_{2i} and the c_i's can be computed from Re $\eta_{i,j}$, a_m and ρ, for instance $c_5=-\rho|a_1|^2/2$ Re $\eta_{2,1}$. In the x_1, x_2 plane let us consider asymptotic directions where $\xi_1=0$. When $\xi_2\to\pm\infty$, we find $|\hat{K}^1_2|\to$ constants. One of the two constants $(c_5/(1+c_1)^2)$ being zero only if $a_1=0$, we come back to our previous $m_0=1$ case. Consequently $|\hat{K}^1_2|$ cannot be confined in the x_1, x_2 plane.

2.4. NLPDE *satisfied by the solutions of equation* (6) *for* $n \geqslant 3$

In order to be an IE, the K_j^i solutions of equation (6) must satisfy the NLPDE (equation (4)). However we shall see for $n \geqslant 3$ that the potentials $\hat{K}_j^i$ and the solutions K_j^i are constrained to satisfy other particular NLPDE. We remark from equation (8) that our kernels F_j^i, $\tilde{F}_j^i$ satisfy:

$$\begin{aligned}&(\lambda_k^{-1}D_0(x_k)-\lambda_q^{-1}D_0(x_q))F_j^i=0\\&(\lambda_k^{-1}D_0(x_k)-\lambda_q^{-1}D_0(x_q))\tilde{F}_j^i=0\qquad k\neq j,\ q\neq j,\ n\geqslant 3.\end{aligned}\tag{16}$$

Consequently from equation (6) we get:

$$(\lambda_k^{-1}D_0(x_k)-\lambda_q^{-1}D_0(x_q))K_j^i$$
$$=\lambda_q^{-1}\tilde{F}_q^i\hat{K}_j^q-\lambda_k^{-1}\tilde{F}_k^i\hat{K}_j^k+\sum_m\int_{x_m}F_m^i(\lambda_k^{-1}D_0(x_k)-\lambda_q^{-1}D_0(x_q))K_j^m$$

and comparing with the solution of the IE (equation (6)):

$$(\lambda_k^{-1}D_0(x_k)-\lambda_q^{-1}D_0(x_q))K_j^i=-\lambda_k^{-1}K_k^i\hat{K}_j^k+\lambda_q^{-1}K_q^i\hat{K}_j^q,\qquad k\neq j,\ q\neq j,\ n\geqslant 3.\tag{17}$$

For $n \geqslant 4$ we can even get NLPDE between the potentials $\hat{K}_j^i$:

$$(\lambda_k^{-1}D_0(x_k)-\lambda_q^{-1}D_0(x_q))\hat{K}_j^i=\lambda_q^{-1}\hat{K}_q^i\hat{K}_j^q-\lambda_k^{-1}\hat{K}_k^i\hat{K}_j^k\tag{18}$$

i, j, k, q being all different $n \geqslant 4$.

Even for $n=3$ where all the i, j, k, q cannot be different we get an NLPDE between the potentials $\hat{K}_j^i$ if between the two relations (4) and (17) we eliminate the $K_i^i\hat{K}_j^i$ term. We get:

$$\left(\frac{\lambda_k}{\lambda_j}D_0(x_j)+\frac{\lambda_k}{\lambda_i}D_0(x_i)-D_0(x_k)\right)\hat{K}_j^i=2\hat{K}_k^i\hat{K}_j^k\qquad i\neq j\neq k\neq i.\tag{19}$$

This equation is similar to a three-wave non-linear evolution equation where one coordinate represents the time whereas the two other span a two-dimensional coordinate space. Let us remark in the examples (14) and (15) that if one coordinate is a time and has fixed values then the $\hat{K}_j^i$ are confined in the plane corresponding to the other two coordinates.

Finally for $n \geqslant 3$, the IE (equation (6)) leads both to non-confined solutions in the n-coordinate space and to potentials restricted to satisfy well defined NLPDE (equations (17), (18) and (19)) and we conclude that it applies only to a subspace of the space of possible potentials associated with the system (1).

3. Inversion equations in some particular (less than N) coordinate case

If in the linear partial system (1) we put all the coordinates x_i equal, we must recover the associated one-coordinate IE. Now what happens if two coordinates are equal, three coordinates, . . . ?

3.1. Statement of the problem

Let us assume in (1) that the first q coordinates are different whereas the $(n-q)$ remaining ones are equal: $x_q = x_{q+1} = \ldots = x_n$.

$$(\Delta_0(x_1, x_2, \ldots, x_q, x_{q+1} = x_q, \ldots, x_n = x_q) + \mathrm{i}k\Lambda - Q)\psi = 0.$$

As above we call $D_0(u)$ the operator $\partial/\partial u$ and the elements of the diagonal Δ_0 matrix are $D_0(x_1), D_0(x_2), \ldots, D_0(x_q), D_0(x_q), \ldots, D_0(x_q)$. Let us consider the representation (2) where $x_j = x_q$ for $j \geqslant q$:

$$\psi_j = \left(\delta_{ij} U_j^0(x_j) + \int_{x_j}^{\infty} U_j^0(y) K_i^j(x_1, x_2, \ldots, x_q; y)\,\mathrm{d}y\right). \tag{2'}$$

If the $\{K_j^i\}$ satisfy (3′) ((3) with $x_m = x_q$ for $m \geqslant q$) and

$$O_{jxj}^i K_j^i = \sum_{m \neq j}^{m=n} K_m^i \hat{K}_j^m \epsilon_j^m, \qquad \hat{K}_j^m = K_j^m(x_1, \ldots, x_q; y = x_m)$$

$$\epsilon_j^m = \begin{cases} \left(\dfrac{\lambda_j}{\lambda_m} - 1\right) & \text{for both } j \geqslant q,\ m \geqslant q \\[2ex] \dfrac{\lambda_j}{\lambda_m} & \text{for either } j < q \text{ or } m < q \end{cases} \tag{4'}$$

$$q_i^i = 0,$$

$$q_j^i = \begin{cases} \left(\dfrac{\lambda_j}{\lambda_i} - 1\right)\hat{K}_j^i & \text{for both } j \geqslant q,\ i \geqslant q \\[2ex] \dfrac{\lambda_j}{\lambda_i}\hat{K}_j^i & \text{for either } j < q \text{ or } i < q \end{cases} \tag{5'}$$

then one can show that the $\{\psi_j\}$ are solutions of equation (1′).

Let us notice that if $q = 1$, equations (4′) and (5′) reduce to the corresponding equations of the one-coordinate case whereas if $q = n$, equations (4′) and (5′) are identical to equations (4) and (5).

3.2. Inversion equations associated with the system (1′)

Let us consider

$$\begin{aligned} K_j^i(x_1, \ldots, x_q; y) &= \tilde{F}_j^i(x_1, \ldots, x_q; y) \\ &\quad + \sum_m^{m=n} \int_{x_m}^{\infty} F_m^i(s; x_1, \ldots, x_q; y) K_j^m(x_1, \ldots, x_q; s)\,\mathrm{d}s \end{aligned}$$

$$\tilde{F}_j^i(x_1, \ldots, x_q; y) = F_j^i(s = x_j; x_1, \ldots, x_q; y) \tag{6'}$$

where $x_m = x_q$ for $q \leqslant m \leqslant n$ and $x_j = x_q$ for $q \leqslant j \leqslant n$. Let us assume that the kernels F_j^i satisfy the boundary condition (7′) ((7) with $x_m = x_q$ for $m \geqslant q$) and q independent LPDE:

$$\lambda_1^{-1} D_0(x_1) F_j^i = \lambda_2^{-1} D_0(x_2) F_j^i = \ldots = \lambda_q^{-1} D_0(x_q) F_j^i = -(\lambda_j^{-1} D_0(s) + \lambda_i^{-1} D_0(y)) F_j^i. \tag{8'}$$

We assume of course that the solution of equation (8) exists and is unique.

Property. If we assume that the $\{F_j^i\}$ satisfy both equations (7′) and (8′) and *further that* $\lambda_q = \lambda_{q+1} = \ldots = \lambda_n$, then the solutions $\{K_j^i\}$ of equation (6′) satisfy the NLPDE (equation (4′)).

From equation (4′) and $\lambda_m = \lambda_q$ for $m \geqslant q$ we get: $\epsilon_j^m = \lambda_j(\lambda_m)^{-1}$ for $j < q$, and for $j \geqslant q$, $\epsilon_j^m = \lambda_j(\lambda_m)^{-1}$ if $m < q$ and $\epsilon_j^m = 0$ if $m \geqslant q$. For the proof, due to equation (8′), $O_{jx_j}^i \tilde{F}_j^i = 0$ and applying $O_{jx_j}^i$ to both sides of equation (6′) we get that

$$O_{jx_j}^i K_j^i - \sum_m \int_{x_m}^{\infty} F_m^i O_{jx_j}^m K_j^m = \begin{cases} \displaystyle\sum_{\substack{m \neq j}}^{m=n} \lambda_j(\lambda_m)^{-1} F_m^i \hat{K}_j^m & \text{if } j < q, \\ \displaystyle\sum_{m=1}^{q-1} \lambda_j(\lambda_m)^{-1} F_m^i \hat{K}_j^m & \text{if } j \geqslant q. \end{cases}$$

Comparing with the IE (equation (6′)), the result (4′) follows. We remark:

(i) From $\lambda_m = \lambda_q$ for $m \geqslant q$ and equation (5′) we get $q_j^i \equiv 0$ for both $i \geqslant q, j \geqslant q$. It follows that if the $\{F_j^i\}$ depend explicitly upon the $x_1, \ldots, x_n$ in a case where $x_m = x_q$ for $m \geqslant q$, then necessarily $\lambda_m = \lambda_q$ for $m \geqslant q$ and the potentials $q_j^i (j \geqslant q, i \geqslant q)$ are zero.

(ii) For $q = 1$, we have $x_1 = x_2 = \ldots = x_n = x$, a one-coordinate case. It follows that if the $\{F_j^i(s; x; y)\}$ depend upon the coordinate x, then necessarily $\lambda_1 = \lambda_2 = \ldots = \lambda_n$ and all the potentials q_j^i are zero. Our IE (6′) fails in the one-coordinate case.

(iii) If the $\{F_j^i(s; y)\}$ are independent of the $x_1, \ldots, x_q$, then equation (8′) reduces to $(D_0(s) + \lambda_j(\lambda_i)^{-1} D_0(y)) F_j^i = 0$, $O_{jx_j}^i \tilde{F}_j^i = 0$, the $\{\lambda_m\}$ for $m \geqslant q$ are not necessarily equal and we recover our previous IE (Cornille 1978).

In conclusion, for the system (1) where two, three, ..., coordinates are equal, there exist a least two different associated IE as long as we have more than one coordinate.

3.3. *Properties of the kernels* $\{F_j^i\}$

From the LPDE (equation (8′)) we see that F_j^i still depends upon two independent variables U_j^i and V_j^i:

$$F_j^i\left(U_j^i = \lambda_j(x_j - s) + \sum_{m \neq j}^{m=q} \lambda_m x_m + \nu_j^i;\ V_j^i = \lambda_i(x_i - y) + \sum_{l \neq i}^{l=q} \lambda_l x_l + \eta_j^i \right), \quad (10')$$

where if $j \geqslant q$, $\lambda_j = \lambda_q$, $m \neq q$, $x_j = x_q$, if $i \geqslant q$, $\lambda_i = \lambda_q$, $l \neq q$, $x_i = x_q$. Because the $q_j^i = 0$ for both $i \geqslant q, j \geqslant q$ or for $i = j$, we can take $F_j^i \equiv 0$ for $i \geqslant q, j \geqslant q$ or $i = j$.

It follows from equation (10′), as in § 2.2, that the degenerate kernels are not reduced to purely exponential forms.

(i) If the F_j^i are exponentials, because s and y appear in the linear combination $\lambda_j s + (\text{constant})\lambda_i y$ we can always choose the constants in such a way that $F_j^i \to 0$ when $s \to \infty$ or $y \to \infty$.

(ii) In the following we consider only degenerate kernels of the same type as equation (11) where $g_j^i \to 0$, $h_j^i \to 0$ when respectively $|U_j^i| \to \infty$, $|V_j^i| \to \infty$. In order to investigate the possibility of confined solutions we first consider $\tilde{F}_j^i(y = x_i)$.

$$\tilde{F}_j^i(y = x_i) = g_j^i\left(\sum_{m \neq j}^{m=q} \lambda_m x_m + \nu_j^i\right) h_j^i\left(\sum_{m \neq i}^{m=q} \lambda_m x_m + \eta_j^i\right) \quad (12')$$

where if $j \geqslant q$, $m \neq q$ and if $i \geqslant q$, $l \neq q$. As previously, in order that $\tilde{F}^i_j(y = x_i)$ can vanish in the R^q space, g^i_j and h^i_j must depend on only one coordinate and so $q = 2$, $n > 2$. Further the two coordinates appearing respectively in g^i_j and h^i_j must be different. We take for $q = 2$, $\tilde{F}^i_j = 0$ if both $i \geqslant 2$, $j \geqslant 2$, or $i = j$ and for the remaining cases

$$\tilde{F}^1_j = g^1_j(\lambda_1 x_1 + \nu^1_j) h^1_j(\lambda_2 x_2 + \eta^1_j), \qquad \tilde{F}^i_1 = g^i_1(\lambda_2 x_2 + \nu^i_1) h^i_1(\lambda_1 x_1 + \eta^i_1)$$

which are confined in the x_1, x_2 plane.

For $q > 2$, our $\tilde{F}^i_j(y = x_i)$ are not confined in the R^q space; however let us consider $q = 3$ and $n > q$. We take $F^i_i = 0$ and F^1_m, F^2_m, F^m_2, F^m_1 as the only degenerate kernels different from zero. We get

$$\tilde{F}^i_j(y = x_i) = g^i_j(\lambda_i x_i + \lambda_k x_k + \nu^i_j) h^i_j(\lambda_j x_j + \lambda_k x_k + \eta^i_j) \qquad \text{where } x_i \neq x_j \neq x_k,\ i \neq j \neq k;$$

if $i = 1$, $j \geqslant 3$ then

$$\lambda_k x_k = \lambda_2 x_2, \qquad \lambda_j x_j = \lambda_3 x_3;$$

if $i = 2$, $j \geqslant 3$ then

$$\lambda_k x_k = \lambda_1 x_1, \qquad \lambda_j x_k = \lambda_3 x_3;$$

if $j = 1$, $i \geqslant 3$ then

$$\lambda_i x_i = \lambda_3 x_3, \qquad \lambda_k x_k = \lambda_2 x_2$$

and if $j = 2$, $i \geqslant 3$ then

$$\lambda_i x_i = \lambda_3 x_3, \qquad \lambda_k x_k = \lambda_1 x_1.$$

We verify that if any one x_k value is fixed, then the $\{\tilde{F}^i_j(y = x_i)\}$ are confined in the associated x_i, x_j plane.

3.4. *Some simple examples for $q = 2$ and $n \geqslant 3$*

We assume that the kernels of equation (6′) are of the form $F^i_j = g^i_j h^i_j$, given by equation (11), and restrict our study to $q = 2$, $n > q$; i.e. a two-dimensional x_1, x_2 space. We consider

$$\mathscr{F} = \begin{pmatrix} 0 & F^1_2\theta(s - x_2) \ldots F^1_n\theta(s - x_2) \\ F^2_1\theta(s - x_1) & \\ \vdots & 0 \\ F^n_1\theta(s - x_1) & \end{pmatrix} \tag{15}$$

$$F^1_j = F^1_j(\lambda_2(x_2 - s) + \lambda_1 x_1 + \nu^1_j;\ \lambda_1(x_1 - y) + \lambda_2 x_2 + \eta^1_j)$$
$$F^i_1 = F^i_1(\lambda_1(x_1 - s) + \lambda_2 x_2 + \nu^i_1;\ \lambda_2(x_2 - y) + \lambda_1 x_1 + \eta^i_1).$$

The potentials $\hat{K}^1_j$ and $\hat{K}^j_1$ ($j = 1, \ldots, n$) are obtained from equation (6′) by the resolution of a linear algebraic system.

(i) For $n=3$, $\mathscr{F}$ in equation (15) reduces to the kernel (15a) and the potentials $\hat{K}^1_2$, $\hat{K}^1_3$, $\hat{K}^2_1$, $\hat{K}^3_1$ are given in a closed form (15b) where (15c) is replaced by

$$\begin{aligned} g^1_j &= g^1_j(\lambda_1 x_1+\nu^1_j), & g^i_1 &= g^i_1(\lambda_2 x_2+\eta^i_1), \\ h^1_j &= h^1_j(\lambda_2 x_2+\eta^1_j), & h^i_1 &= h^i_1(\lambda_1 x_1+\eta^i_1) \end{aligned} \qquad (15'c)$$

$$A^i_{j,k}=\int_0^\infty g^i_k(-\lambda_k u+\lambda_i x_i+\nu^i_k)h^k_j(-\lambda_k u+\lambda_i x_i+\eta^k_j)\,\mathrm{d}u$$

$\lambda_k=\lambda_2$ if $i=1$ and $\lambda_k=\lambda_1$, $\lambda_i=\lambda_2$ if $i>1$. From (15b), (15$'c$) we see that $A^i_{j,k}$ and D are bounded when $\rho=(x_1^2+x_2^2)^{1/2}\to\infty$. Assuming, as always in the discussion that $D\neq 0$, for the $\hat{K}^i_j$ in (15b) factors like $g^l_m h^{l'}_m$ or $g^l_m h^l_{m'}$ appear, which vanish when $\rho\to\infty$.

(ii) For $n>3$, even if the solution is too complicated to be written in closed form, one can explicitly show the confinement properties. The potentials are $\hat{K}^1_j$ and $\hat{K}^i_1$ ($j=2,\ldots,n$). We easily get from equation (6′) for $\hat{K}^1_j$:

$$\hat{K}^1_j=g^1_j\Big(h^1_j+\sum_p h^1_p A^1_{1p}B^p_j\Big)$$

$$B^p_j(1-A^p_{p1}A^1_{1p})-\sum_{q\neq p}B^q_j A^p_{q1}A^1_{1q}=A^p_{j,1}$$

and for $\hat{K}^i_1$:

$$\begin{cases} \hat{K}^i_1=h^i_1\Big(g^i_1+\sum_p A^i_{p1}C_p\Big) \\ C_p(1-A^1_{1p}A^p_{p1})-\sum_q A^1_{1p}A^p_{q1}C_q=A^1_{1p}g^p_1 \end{cases}$$

where h^i_j, g^i_j and $A^i_{j,k}$ are still given by (15$'c$). Always including the zeros of the Fredholm determinants, we see that $A^i_{j,k}$, B^p_j, C_p are bounded in R^2. For $\hat{K}^1_j$ the factors $g^1_j h^1_j$, $g^1_j h^1_p$ appear and for $\hat{K}^i_1$ the factors $h^i_1 g^i_1$, $h^i_1 g^p_1$ appear (for the term where C_p is present) which vanish when $p\to\infty$.

So for $q=2$, $n>2$, the potentials reconstructed from equation (15) ($F^i_j=g^i_j h^i_j$ being of the equation (11) type) are confined in the x_1, x_2 plane.

3.5. *NLPDE satisfied by the solutions of equation* (6′) *for* $q\geqslant 3$, $n\geqslant 4$

For $n\geqslant 4$, $q\geqslant 3$ and from equation (8′) we remark that the operator $\lambda_k^{-1}D_0(x_k)-\lambda_p^{-1}D_0(x_p)$ applied to F^i_j or $\tilde{F}^i_j$ gives zero (j,k,p, all different and x_j,x_k,x_p all different). If we apply the same operator to both sides of equation (6′) and compare with the solution of equation (6′) we get:

$$\begin{aligned} &\lambda_k^{-1}D_0(x_k)-\lambda_p^{-1}D_0(x_p))K^i_j=-(\lambda_k)^{-1}K^i_k\hat{K}^k_j+(\lambda_p)^{-1}K^i_p\hat{K}^p_j \qquad \text{if } x_k\neq x_q,\ x_p\neq x_q \\ &\lambda_q^{-1}D_0(x_q)-\lambda_p^{-1}D_0(x_p))K^i_j=(\lambda_p)^{-1}K^i_p\hat{K}^p_j-\sum_{m=q}^{m=n}K^i_m\hat{K}^m_j(\lambda_q)^{-1}. \end{aligned} \qquad (17')$$

If x_i, x_j, x_k, x_p are all different (which means necessarily $q\geqslant 4$), the relations (17′) hold for the potentials; i.e. when K^i_j, K^i_p, K^i_m are replaced by $\hat{K}^i_j$, $\hat{K}^i_p$, $\hat{K}^i_m$.

However even for $q=3$, $n>3$, i.e. an R^3 space, we obtain NLPDE satisfied by the potentials $(\hat{K}^1_j, \hat{K}^2_l, \hat{K}^j_1, \hat{K}^l_2;\ j=2,\ldots,n;\ l=3,\ldots,n)$ if into the relations (17) we take into account the NLPDE (equation (4′)). We get:

$$\begin{aligned}
&(\lambda_j^{-1}D_0(x_j)+\lambda_i^{-1}D_0(x_i)-\lambda_3^{-1}D_0(x_3))\hat{K}^i_j \\
&\qquad =2\lambda_3^{-1}\sum_{m=3}^{m=n}\hat{K}^i_m\hat{K}^m_i; \qquad (i,j)=(1,2)\text{ and }(2,1) \\
&(\lambda_j^{-1}D_0(x_j)+\lambda_3^{-1}D_0(x_3)-\lambda_k^{-1}D_0(x_k))\hat{K}^i_j \\
&\qquad =2\lambda_k^{-1}\hat{K}^i_k\hat{K}^k_j; \qquad i=3,\ldots,n;\ (j,k)=(1,2)\text{ and }(2,1) \\
&(\lambda_j^{-1}D_0(x_j)+\lambda_3^{-1}D_0(x_3)-\lambda_k^{-1}D_0(x_k))\hat{K}^j_i \\
&\qquad =2\lambda_k^{-1}\hat{K}^j_k\hat{K}^k_i; \qquad i=3,\ldots,n;\ (j,k)=(1,2)\text{ and }(2,1).
\end{aligned} \tag{19'}$$

These NLPDE are generalised non-linear three-wave equations if one coordinate is a time and the two other define an R^2 space. If we assume $F^i_j=g^i_jh^i_j$ of the equation (11) type, then with straightforward but tedious calculations we could verify that at fixed time the potentials are confined in this two-coordinate plane. However in the next section we shall study explicitly a similar simpler case $q=3$, $n=3$.

3. Confined solutions of the three-wave NLPDE in a two-dimensional space

In the previous sections we have seen for $q\geqslant 3$ that both the potentials associated to our IE are not confined in the $x_1,\ldots,x_q$ space and that they must satisfy a generalised three-wave NLPDE if one coordinate is interpreted as a time. Here for $q=3$ (and only $n=3$ for simplicity) we study further this NLPDE and particularly the confinement problem in a two-dimensional space. We come back for $n=q=3$ to our example $\mathcal{F}$ given by (15a) ($F^i_j=g^i_jh^i_j$ being of the same type as equation (11)) where the six potentials $\hat{K}^i_j$, i (or j) $=1,2,3$, $i\neq j$ satisfy equation (19):

$$O_k\hat{K}^i_j=2\hat{K}^i_k\hat{K}^k_j, \qquad O_k=\lambda_k\lambda_j^{-1}\frac{\partial}{\partial x_j}+\lambda_k\lambda_i^{-1}\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_k}.$$

We remark that $\hat{K}^i_j$ and $\hat{K}^j_i$ satisfy the same NLPDE. If the functions g^i_j and h^i_j are linked then $\hat{K}^i_j$ and $\hat{K}^j_i$ are also linked. Let us assume:

$$g^i_j=\sigma^i_j(h^j_i)^*, \qquad \sigma^i_j \text{ real numbers.} \tag{20}$$

Then from equations (15b,c) we get:

$$\hat{K}^2_3=\sigma^2_1\sigma^1_3(\sigma^3_1\sigma^1_2)^{-1}(\hat{K}^3_2)^*, \qquad \hat{K}^2_1=(\hat{K}^1_2)^*\sigma^2_1(\sigma^1_2)^{-1}, \qquad \hat{K}^3_1=(\hat{K}^1_3)^*\sigma^3_1(\sigma^1_3)^{-1}$$

and the NLPDE (equation (19)) can be written between only three potentials. For instance let us choose $B_1=\hat{K}^2_3$, $B_2=\hat{K}^1_3$, $B_3=\hat{K}^2_1$ then $\{B_1,B_2,B_3\}$ satisfy the NLPDE

$$\begin{aligned}
&O_1B_1=2B_2B_3 \\
&O_2B_2=2B_1B_3^*\sigma^1_2(\sigma^2_1)^{-1} \qquad O_k=\frac{\lambda_k}{\lambda_i}\frac{\partial}{\partial x_i}+\frac{\lambda_k}{\lambda_j}\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_k} \\
&O_3B_3=2B_1B_2^*\sigma^3_1(\sigma^1_3)^{-1}.
\end{aligned} \tag{21}$$

Taking into account the relation of equation (20) in (15*b*) we eliminate the g_j^i functions and obtain for the B_i's expressions where only the h_j^i appear:

$$DB_3 = \sigma_1^2 h_1^2 \Big[(h_2^1)^* + \sigma_3^1 \sigma_1^3 \int |h_1^3|^2 \, du \Big((h_3^1)^* \int (h_2^{1*} h_3^1) \, du - (h_2^1)^* \int |h_3^1|^2 \, du \Big) \Big]$$

if $2 \leftrightarrow 3$, then $D(\sigma_1^2)^{-1} B_3 \leftrightarrow D(\sigma_3^1)^{-1} B_2^*$

$$DB_1 = \sigma_3^1 \sigma_1^2 h_1^2 (h_1^3)^* \int (h_2^{1*} h_3^1) \, du$$

$$D = 1 - \sigma_2^1 \sigma_1^2 \int |h_1^2|^2 \, du \int |h_2^1|^2 \, du - \sigma_1^3 \sigma_3^1 \int |h_1^3|^2 \, du \int |h_3^1|^2 \, du \qquad (22)$$

$$+ \sigma_2^1 \sigma_1^2 \sigma_3^1 \sigma_1^3 \int |h_1^2|^2 \, du \int |h_1^3|^2 \, du \Big(\int |h_2^1|^2 \, du \int |h_3^1|^2 \, du - \Big| \int h_2^{1*} h_3^1 \, du \Big|^2 \Big)$$

where $h_j^i = h_j^i(\lambda_j x_j + \lambda_k x_k + \eta_j^i)$

$$\int h_j^{i*} h_k^i \, du = \int_0^\infty (h_j^i(-\lambda_i u + \lambda_j x_j + \lambda_k x_k + \eta_j^i))^* h_k^i(-\lambda_i u + \lambda_j x_j + \lambda_k x_k + \eta_k^i) \, du.$$

Finally we have four independent arbitrary functions $h_2^1, h_3^1, h_1^2, h_1^3$ that we take to be of the equation (11) type. Firstly, if we construct B_1, B_2, B_3 following the above formulae we can directly verify that they satisfy the NLPDE (21). Secondly if we consider one of the three variables x_1, x_2, x_3 as a fixed time, we can verify (outside the values where the Fredholm determinant D is zero) that the B_i's are confined in the plane corresponding to the other two variables. Applying the Schwarz inequality to the third term of D and if $\sigma_2^1 \sigma_1^2 < 0$, $\sigma_3^1 \sigma_1^3 < 0$, let us remark that in this case $D \neq 0$ whereas in other cases D can have zeros.

4. Conclusion

In this paper we were asking if there exist potentials reconstructed from the inversion formalism which are really confined in a more than one coordinate space. We have determined IE associated with an $n \times n$ system of linear partial differential equations with q different coordinates $(1 < q \leq n)$. When $q > 1$ we find that there exist at least two different IE, the second being a generalisation of the first.

For the first IE, the kernels F_j^i, which are almost independent of the coordinates, depend upon one variable $U_j^i(s, y)$, and the pure exponential degenerate kernels cannot lead to confined reconstructed potentials. In the following we shall always discuss the second IE.

For the second IE the kernels F_j^i depend upon two independent variables $U_j^i(s, x_1, \ldots, x_q)$, $V_j^i(y, x_1, \ldots, x_q)$ and the degenerate kernels $g_j^i(U_j^i) h_j^i(V_j^i)$ (or a finite sum of such terms) are not restricted to be purely exponential. For $q = 2$ we show that there exist confined reconstructed potentials in the x_1, x_2 plane and they are not constrained to satisfy particular NLPDE. On the contrary for $q \geq 3$, both the reconstructed potentials are not confined in the corresponding R^q space and they must satisfy NLPDE similar to the generalised non-linear three-wave evolution equation in an R^{q-1} space if we interpret one coordinate as a time. So for $q \geq 3$, our IE applies on a subspace of the space of all the potentials associated with our starting linear partial differential system. We think that there could exist for $q \geqslant 3$ other IE where these drawbacks disappear.

However, if for $q = 3$ we interpret one coordinate as a time, the non-linear three-wave evolution equations exhibit an infinite number of explicit solutions which, for any finite time, are confined in a two-dimensional coordinate space. In order to see the progress obtained we can compare with the old Zakharov and Shabat (1974) results†. For instance here, in the most simple degenerate case (when the kernels are simply a product of a function of y by a function of s), we get explicit solutions which can be considered as solitary waves if we adopt the definition recently given by Strauss (1977).

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† From the talk given by Novikov (1977) we are becoming aware that Zakharov and Shabat have recently obtained confined NLPDE solutions in a two-dimensional space case.